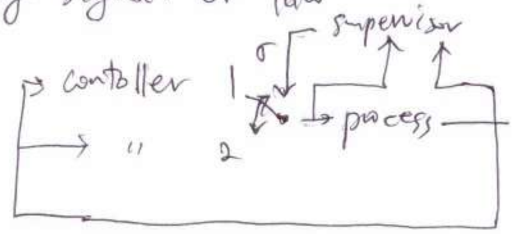
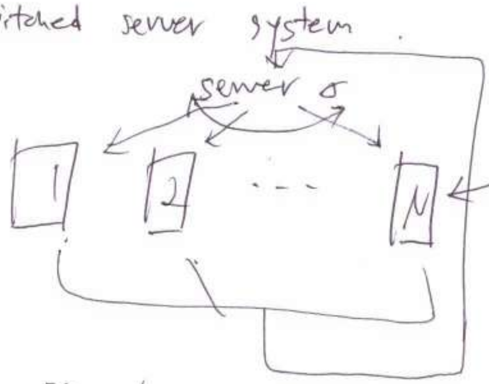
§Ch1. Motivating Examples.

1.1 Switched Systems → switching signal or law

- Supervisory switching control



- switched server system.



$\sigma : \mathbb{R}^n \to \{1, 2, \cdots, N\}$.

- singular system with Markov switching $\{r(t), t \geq 0\}$: Markov process.
  $r : \mathbb{R}^+ \to \{1, 2, \cdots, N\}$.

1.2 Impulsive Systems

- SEIRS Model
  $S(t)$: susceptible $E(t)$: exposed $I(t)$: infectious $R(t)$: recovered.
  $N(t) = S(t) + E(t) + I(t) + R(t)$.
  Saturation incidence (hospital beds) or vaccination ⇒ time delay or impulsive effects.

pg 4. equation: time delay and jumps (t^+)

- Insulin treatment (food ⇒ sugar level continuously. insulin ⇒ jump)

§Ch2. Mathematical Background

2.1 Basic Definitions

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$ IVP. WLOG, $f(t, 0) = 0$ so that $x \equiv 0$: equilibrium.

Def 2.1

$x \equiv 0$ is (i) stable if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0$.

(ii) uniformly stable if stable + δ : indep of t_0

(iii) asymptotically stable if (i) + $\exists c \overset{= c(t_0)}{>} 0$ s.t. $\|x_0\| < c \Rightarrow \lim_{t \to \infty} x(t) = 0$.

(iv) uniformly asymptotically stable (ii) + c : indep of t_0.

(v) exponentially stable $\|x(t)\| \leq k \|x_0\| e^{-\lambda(t - t_0)} \quad \forall \|x_0\| < c$.

(vi) unstable if (i) does not hold.

Def 2.2 $V: D \to \mathbb{R}$ is positive semi-definite if i) $V(t,0)=0$
ii) $V(t,x) \geq 0 \quad \forall x \in D\setminus\{0\}$.

positive definite if ii) $V(t,x) > 0$

radially unbounded : positive definite & $\lim_{\|x\|\to\infty} V(t,x) = \infty \quad \forall t$.

Def 2.3 $D^+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}\left[V(t+h, x+hf(t,x)) - V(t,x)\right]$

Dini derivative.

$\dot{V}(t,x) = \frac{\partial V(t,x)}{\partial t} + \nabla_x V(t,x)\cdot f(t,x)$.

To guarantee stability, need special class of ftns. ex) K, KL, $K_1, K_2, \cdots$ $K_c, K_3, K_V, \cdots$

Thm 2.1 W_1, W_2, W_3 : pos. def. V satisfies i) $W_1(x) \leq V(t,x) \leq W_2(x)$
ii) $\dot{V}(t,x) = \frac{\partial V(t,x)}{\partial t} + \nabla V_x(t,x)\cdot f(t,x) \leq 0$.

$\Rightarrow x$: uniformly stable.

If $\dot{V}(t,x) \leq -W_3(x) \quad \forall (t,x)$, then $x\equiv 0$ uniformly asymptotically stable

If $\exists\, r, c$ s.t. $\min_{\|x\|=r} W_1(x) > c$, for $\forall x$ starts in B_r s.t. $W_2(x) \leq c$.

we have $\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0)$, $\beta \in KL$

W_1 : radially unbounded $\Rightarrow x\equiv 0$ globally uniformly asymptotically stable.

Note KL: $\beta(\cdot, s) \in K$, $\beta(r,\cdot)$: decreasing & $\beta(r,s)\to 0$ as $s\to\infty$

K : $\alpha \in K$ if α is strictly increasing & $\alpha(0)=0$.

- ISS (Input-to-State stable).

$$(*)\begin{cases} \dot{x} = f(t,x,u) & t \geq t_0 \\ x(t_0) = x_0 \end{cases}$$

u : input, perturbation, $u \in PC(\mathbb{R}_+ : \mathbb{R}^n)$ (i.e. piecewise continuous), $\sup_{t\geq t_0} \|u(t)\| < \infty$.

Def 2.11 $(*)$ is ISS if $\exists\, \beta \in KL$ & $\gamma \in K$ s.t. $\forall x_0$, bdd u,
$x(t)$ exists and $\|x(t)\| \leq \beta(\|x_0\|, t-t_0) + \gamma\left(\sup_{t_0\leq s\leq t} \|u(s)\|\right) \quad \forall t \geq t_0$

Rmk $u\equiv 0 \Rightarrow$ ISS becomes globally uniformly asymptotic stable.

Thm 2.2 $a, b \in K_\infty$ ($a, b \in K$ & $a(r), b(r) \to \infty$ as $r\to\infty$)
$\rho \in K$, c : pos def
V satisfies
$$\begin{cases} b(\|x\|) \leq V(t,x) \leq a(\|x\|) & \forall (t,x) \\ \dot{V}(t,x,u) \leq -c(x) & \forall \|x\| \geq \rho(\|u\|) \end{cases}$$
for $\forall (t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$

$\Rightarrow$ System is ISS with $\gamma(\cdot) = b^{-1}(a(\rho(\cdot)))$.

2.2 Comparison Method

Thm 2.3 $\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$. $V: \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$ conti, ~~....~~ loc Lipschitz in x.

Assume V satisfies $D^+V(t, x) \le g(t, V(t, x))$ where g: conti.

Let $r(t) = r(t : t_0, u_0)$ be maximal solution of $\begin{cases} \dot{u} = g(t, u) \\ u(t_0) = u_0 \ge 0 \end{cases}$

Then $V(t_0, x_0) \le u_0$ implies $V(t, x(t)) \le r(t)$ $\forall t \ge t_0$

$f(t, x) \le g(t, x)$.
$\Rightarrow V := x$, $\Rightarrow f(t, x) \le g(t, x)$
$\Rightarrow x(t) \le r(t)$

Thm 2.4 $a, b \in K$, V: loc Lip in x, (i) $b(\|x\|) \le V(t, x) \le a(\|x\|)$ $\forall (t, x)$

(ii) $D^+V(t, x) \le g(t, V(t, x))$

where g: conti & $g(t, 0) = 0$ $\forall t$.

Then stability property of $u \equiv 0$ implies stab prop of $x \equiv 0$.

2.3 Delay Systems

$C_r = C([-r, 0] : \mathbb{R}^n)$.

Def 2.12 Let $t^* \in \mathbb{R}$, $a > 0$. If x is a ftn mapping $[t^* - r, t^* + a] \longrightarrow \mathbb{R}^n$, for each $t \in [t^*, t^* + a]$, define $x_t : [-r, 0] \to \mathbb{R}^n$ by $x_t(s) = x(t+s)$

$\|x_t\|_r = \sup_{t-r \le \theta \le t} \|x(\theta)\|$

$\begin{cases} \dot{x}(t) = f(t, x_t) \\ x_{t_0} = \phi(s) \end{cases}$: functional differential equation, f: functional operator.

$\leftarrow$ IVP of delay-type system.

$s = -r$, $x_t(s) = x(t - r)$: delay system.

Def 2.13, 2.14 skip.

Thm 2.5 stability of time-delay (add some conditions such as $V(t+\theta, \psi(\theta)) \le V(t, \psi(0)) \ldots$)

$\uparrow$ time-delayed ftn.

* Important Example

$\dot{x}(t) = Ax(t) + Bx(t - r)$ A, B : matrices r: time-delay.

$\Rightarrow$ One encounters $\dot{v}(t) \le -\alpha V(t) + \beta \sup_{t-r \le s \le t} V(s)$, $\alpha, \beta > 0$

Upper bound of $V(t)$.

Lemma 2.1 $\alpha > \beta > 0 \Rightarrow \exists \gamma > 0, k > 0$ s.t. $V(t) \le k e^{-\gamma(t - t_0)}$ where $-\gamma = -\alpha + \beta e^{\gamma r}$

$k = \inf_{t_0 - r \le s \le t_0} V(s)$

나머지 Lemma : skip.

2.4 Impulsive Systems

$\dot{x}(t) = f(t, x(t))$ $t \neq \tau_k(x)$ | $\Delta x(t) = x(t^+) - x(t)$, $x(t+) = \lim_{\varepsilon \to 0+} x(t+\varepsilon)$.

$\Delta x(t) = F(t, x(t))$ $t = \tau_k(x)$ | $x(t) = x(t-)$ (LC) .

$x(t_0^+) = x_0$.

$0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \cdots$, $\lim_{k\to\infty} \tau_k(x) = \infty$.

$\tau_k(x)$: impulse time. (finite or countable)

if τ_k : constants,

then all solution undergo impulses at same time.

At $t = \tau_k(x)$, $x(t+) = x(t) + F(t, x(t))$.

To have RC sol,
$$\begin{cases} \dot{x}(t) = f(t, x(t)) & t \neq \tau_k(x(t-)) \\ \Delta x(t) = F(t, x(t-)) & t = \tau_k(x(t-)) \\ x(t_0) = x_0 \end{cases}$$

$$x = x(t : t_0, x_0) = \begin{cases} x(t : t_0, x_0), & t_0 < t \le t_1 \\ x(t : t_1, x(t_1^+)), & t_1 < t \le t_2 \\ \vdots \\ x(t : t_k, x(t_k^+)), & t_k < t \le t_{k+1} \\ \vdots \end{cases}$$

Stability: only for $\tau_k(x) = \tau_k$ $\forall x$.

Def 2.16 skip

$$(2.24) \quad \begin{cases} \dot{x}(t) = Ax(t) & t \neq t_k \quad k \in \mathbb{N} \\ \Delta x(t) = B_k x(t) & t = t_k \\ x(t_0^+) = x_0 \end{cases}$$

(A1) $\exists\, 0 < \rho_1 \le \rho$ s.t. $\forall \tau_k \in \mathbb{R}_+$, $\|x(\tau_k^-)\| < \rho_1 \Rightarrow \|x(\tau_k)\| < \rho$.

x defined on $PC(\mathbb{R}_+, D)$, D open.

(A2) $\forall k \in \mathbb{N}$, $\tau_{sup} = \sup\{\tau_k - \tau_{k-1}\} < \infty$

$\tau_{inf} = \inf\{\tau_k - \tau_{k-1}\} > 0$.

Thm 2.6 A: Hurwitz. (2.24) is globally exponentially stable if

$\ln \alpha_k - \nu(t_k - t_{k-1}) \le 0$ $\forall k \in \mathbb{N}$, $\alpha_k = \dfrac{\lambda_{max}\left([I+B_k]^T P (I+B_k)\right)}{\lambda_{min}(P)}$

$\alpha_k e^{-\nu(t_k - t_{k-1})} \le 0$

$0 < \nu < \xi$, $\xi = \dfrac{\lambda_{min}(Q)}{\lambda_{min}(P)}$ Q: pos. def. P: pos def, $A^T P + PA = -Q$.

Pf) $V(x) = x^T P x$, $V(t) = V(x(t))$ $\frac{\partial}{\partial x} V(x) = x^T (P^T + P)$.

$\Rightarrow \dot{V}(t) \le -\xi V(t)$, $t \in (t_{k-1}, t_k]$, $\xi = \lambda_{min}(Q) / \lambda_{max}(P)$.

$V(t) \le V(t_{k-1}^+) e^{-\xi(t - t_{k-1})}$ $t \in (t_{k-1}, t_k]$.

$V(t_k^+) = x(t_k^+)^T P x(t_k^+) = x(t_k)^T (I + B_k)^T P (I + B_k) x(t_k)$

$\le \lambda_{max}\left((I+B_k)^T P (I+B_k)\right) x(t_k)^T x(t_k) = \alpha_k V(t_k)$.

$V(t_k) = x(t_k)^T P x(t_k) \ge \lambda_{min} x(t_k)^T x(t_k)$.

*Hurwitz

$$\begin{pmatrix} a_1 & a_3 & a_5 & & \\ a_0 & a_2 & a_4 & & \\ & a_1 & a_3 & & \\ & & a_2 & & \\ & & a_1 & \cdots & a_n \\ & & & & a_{n-1} \\ & & & & a_{n-2} \; a_n \end{pmatrix}$$

$V(t) \le V(t_0^+) e^{-\xi(t-t_0)} \quad t \in (t_0, t_1]$

$V(t_1^+) \le \alpha_1 V_1(t_1) \le \alpha_1 V(t_0^+) e^{-\xi(t-t_0)}$.

$V(t) \le V(t_0^+) \alpha_1 e^{-\xi(t_1-t_0)} e^{-\xi(t-t_1)} = V(t_0^+)\alpha_1 e^{-\xi(t-t_0)} \quad t \in (t_1, t_2]$.

Generally, for $t \in (t_k, t_{k+1}]$, $V(t) \le V(t_0^+)\alpha_1\alpha_2 \cdots \alpha_k e^{-\xi(t-t_0)}$ $e^{-\nu(t-t_0)} e^{-(\xi-\nu)(t-t_0)}$

$= V(t_0^+)\alpha_1\alpha_2 \cdots \alpha_k \cancel{e^{-\nu(t_1-t_0)} e^{-\nu(t_2-t_1)}}$

$= V(t_0^+)\alpha_1 e^{-\nu(t_1-t_0)} \cdots \alpha_k e^{-\nu(t_k-t_{k-1})} e^{-(\xi-\nu)(t-}$

$\le V(t_0^+) e^{-(\xi-\nu)(t-t_0)} \quad t \ge t_0$.

$\Rightarrow \|x(t)\| \le k \|x(t_0^+)\| e^{-(\xi-\nu)(t-t_0)/2}$

$\left(\lambda_{min}(P)\|x(t)\|^2 \le x(t)^T P x(t) \le x(t_0^+)^T P x(t_0^+) e^{-(\xi-\nu)(t-t_0)} \le \lambda_{max}(P)\|x(t_0)\|^2 e \cdots \right)$

$k = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}}$.

2.5 Comparison Method for Impulsive Systems

$x(t)$: solution of

$$\begin{cases} \dot{x}(t) = f(t,x) & t \ne t_k \\ \Delta x(t) = F_k(x), & t = t_k \\ x(t_0^+) = x_0 & t_0 \ge 0 \end{cases}$$

Thm 2.7 (Comparison Thm)

V: conti in $(t_{k-1}, t_k] \times \mathbb{R}^n$, $\lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+, x)$ loc Lipschitz in x.

~~D+(V(~~ $D^+ V(t,x) \le g(t, V(t,x)) \quad t \ne t_k$,

$V(t, x + F(t,x)) \le \psi_k(V(t,x)) \quad t = t_k$.

g: conti in $(t_{k-1}, t_k] \times \mathbb{R}_+$, $\lim_{(t,y)\to(t_k^+,x)} g(t,y) = g(t_k^+, x)$, ψ_k: nondecreasing.

$r(t)$: max sol of

$$\begin{cases} \dot{u}(t) = g(t,u) & t \ne t_k \\ \Delta u(t) = \psi_k(u) & t = t_k \\ u(t_0^+) = u_0 \ge 0 & t_0 \ge 0 \end{cases}$$

Then,

$V(t_0^+, x_0) \le u_0$ implies $V(t, x(t)) \le r(t) \quad \forall t \ge t_0$.

Thm 2.8 (Stability Thm) ...

2.6 Impulse Systems with time delay

$PC[a,b]$.

read ...

2.7 Stochastic Differential Equations

$X_n \to X$ a.s. : $P(\{\omega \mid \lim_{k\to\infty} X_k(\omega) = X(\omega)\}) = 1$.

in prob : $\forall \varepsilon > 0$, $\lim_{k\to\infty} P(\{\omega \mid |X_k(\omega) - X(\omega)| > \varepsilon\}) = 0$.

in L^p : $\lim_{k\to\infty} E|X_k - X|^p = 0$.

in distribution : $\lim_{k\to\infty} F_k(x) = F(x)$ $\quad F_k, F$: distribution ftn.

$F(x) = P(\{\omega \mid X(\omega) \le x\})$

a.s. $\Rightarrow$ in prob $\Rightarrow$ in dist.
$L^r \nearrow$

stochastic process : $\{X_t(\omega) : t \in I, \omega \in \Omega\}$.

filtration $(F_t)_{t\ge 0}$: $F_t \subset F_s$ for $t \le s$.

$\{F_t\}_{t\ge0}$ is right conti if $F_t = \bigcap_{s>t} F_s$.

usual conditions = right conti, F_0 contains all P-null sets.

$\{X_t\}$: conti = continuous for almost all $\omega \in \Omega$.

cadlag = right-conti, $\exists \lim_{s \nearrow t} X_s(\omega)$ ~~a.e.~~ a.s.

F_t-adapted : X_t is F_t m'ble

X_t, Y_t are indistinguishable if $P(\{\omega \mid X_t(\omega) = Y_t(\omega), \forall t \in \mathbb{R}_+\}) = 1$.

D : open subset. first exit time $\tau = \inf\{t \in \mathbb{R}_+ \mid X(t) \notin D\}$.

Def 2.23 moment

$m_k(t) = E[X^k(t)]$, $\quad Var(X(t)) = m_2(t) - m_1^2(t)$.

Def 2.24 (Ω, F, P) prob space.

(W_t) : Wiener process if
(i) conti
(ii) $P(W_0 = 0) = 1$.
(iii) $\forall 0 \le s < t$, $W_t - W_s \perp F_s$.
(iv) ~~$W_t - W_s \sim N(0, t-s)$~~ $E(W_{t+h} - W_t) = \mu h$
$E((W_{t+h} - W_t)^2) = \sigma^2 h$.

$\mu = 0, \sigma^2 = 1 \Rightarrow$ standard Wiener process.

autocorrelation of X : $R(t_1, t_2) = E(X_{t_1} X_{t_2}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, t_1; x_2, t_2)\, dx_1 dx_2$

$X(t)$: stationary iff $\forall t_1, \cdots, t_n, \tau$, $f_{X(t_1),\cdots X(t_n)}(x_1, \cdots, x_n) = f_{X(t_1+\tau)\cdots X(t_n+\tau)}(x_1, \cdots, x_n)$

Def 2.25 A stochastic process N is Gaussian white noise process iff it is a stationary Gaussian process with mean zero and
$R(\tau) = C\delta(\tau)$, δ : Dirac delta
$= E(X_{t+\tau} X_t)$

Rmk $Var(N(t)) = \infty$.